

Quantum brachistochrone problem for spin-1 in a magnetic field

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We study quantum brachistochrone problem for the spin-1 system in a magnetic field of a constant absolute value. Such system gives us a possibility to examine in detail the statement of papers [A. Carlini *et al.*, Phys. Rev. Lett. **96**, 060503 (2006), D. C. Brody, D. W. Hook, J. Phys. A **39**, L167, (2006)] that *the state vectors realizing the evolution with the minimal time of passage evolve along the subspace spanned by the initial and final state vectors*. Using explicit example we show the existence of quantum brachistochrone with minimal possible time, but the state vector of which, during the evolution *leaves* the subspace spanned by the initial and final state vectors. This is the result of the choice of more constrained Hamiltonian then assumed in the general quantum brachistochrone problem, but what is worth noting, despite that such evolution is more complicated it is still time optimal. This might be important for experiment, where general Hamiltonian with the all allowed parameters is difficult to implement, but constrained one depending on magnetic field can be realized. However for pre-constrained Hamiltonian not all final states are accessible. Present result does not contradict general statement of the quantum brachistochrone problem, but gives new insight how time optimal passage can be realized.

Recently Carlini *et al.* [1] generalized the classical brachistochrone problem to the quantum case. The quantum brachistochrone problem can be formulated in the following way:

What is the optimal Hamiltonian, under a given set of constraints such that the evolution from a given initial state $|\psi_i\rangle$ to given final one $|\psi_f\rangle$ is achieved in the shortest time. Using the variational method the authors solved this problem for some specific examples of constraints which lead to fixed distance between the largest and the smallest energy levels of the Hamiltonian. In [2] it was shown that quantum brachistochrone problem can be solved more directly using the symmetry properties of the quantum state space. That paper was based on the idea considered in [3], where an elementary derivation was provided for passage time from the one quantum state into another orthogonal one.

Later the variational method was extended to allow finding the time-optimal realization of a target unitary operation, when the available Hamiltonians are subjected to certain constraints dictated either by experimental or theoretical conditions [4]. In [5] the authors considered the brachistochrone problem for quantum evolution of mixed states. Very recently Bender *et al.* studied the brachistochrone problem for a PT-symmetric non-Hermitian two-dimensional matrix Hamiltonian [6] and showed that among non-Hermitian PT-symmetric Hamiltonians satisfying the same energy constraint the time evolution between two fixed states can be made arbitrarily small. Such an interesting phenomenon was observed also for dissipative systems described by a non-Hermitian Hamiltonian which has a negative imaginary part of eigenvalues [7]. Discussion on this subject can be found also in [8, 9].

Important statement of work [1, 2] (see also [6]) is that *finding the minimal time of general evolution reduces to finding optimal time evolution for the Hamiltonian acting on the two-dimensional subspace spanned by the initial and final state vectors $|\psi_i\rangle$ and $|\psi_f\rangle$* . It means that optimal evolution which realizes quantum brachistochrone can be written as a linear combination of $|\psi_i\rangle$ and $|\psi_f\rangle$ with time dependent coefficients. One of the aims of our paper is to examine this statement in detail within the three-dimensional quantum system.

We consider the brachistochrone problem in the case when optimal Hamiltonian belongs to the pre-constrained class of Hamiltonians, less general, with a less than allowed number of free parameters that can be used for the problem. Such case is important from the physical point of view when an experimentalist has a possibility to change only a few parameters of Hamiltonian but not all. As an example of such a scenario we consider a three level system, namely, spin-1 in the external magnetic field described by the Hamiltonian of the following

form

$$H = \hbar\omega(\mathbf{n} \cdot \mathbf{s}), \quad (1)$$

where \mathbf{s} are dimensionless spin-1 operators, \mathbf{n} is the direction of the magnetic field and $\hbar\omega$ is proportional to the strength of the magnetic field. Eigenvalues of this Hamiltonian are $-\hbar\omega$, $\hbar\omega$, and 0. The difference between the largest and the smallest eigenvalues is fixed $\Delta E = \hbar\Delta\omega = 2\hbar\omega$, what corresponds to fixed absolute value of magnetic field.

The Hamiltonian (1) contains only two free parameters, namely, two angles θ and ϕ which set the direction of the magnetic field

$$n_x = \sin\theta \cos\phi, \quad n_y = \sin\theta \sin\phi, \quad n_z = \cos\theta. \quad (2)$$

Note that general Hamiltonian in three-dimensional space can be represented by a 3×3 Hermitian matrix which contains nine free parameters (eight if we consider $su(3)$ case). We consider here the pre-constrained class of Hamiltonians (1) with only two free parameters.

The brachistochrone problem in this restricted case reads: What is the optimum choice of the pre-constrained Hamiltonian, namely, what is the optimal direction of the magnetic field \mathbf{n} at fixed ω , such that the evolution from a given initial state $|\psi_i\rangle$ to a given final one $|\psi_f\rangle$ is achieved in the shortest time? Obviously, with such a restriction of possible evolutions not all general final states can be reached. This is the price for taking the narrower family of Hamiltonians. However, as we show it below the shortest time achieved in optimal evolution is the same as in the general setting despite the fact that evolution in our case is more complicated (system leaves the subspace spanned on the initial and final states). Let us see it in detail.

The vector of state for spin-1 contains four parameters. We can write

$$|\psi\rangle = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} |a|e^{i\alpha_1} \\ |b|e^{i\alpha_2} \\ |c|e^{i\alpha_3} \end{pmatrix} = e^{i\alpha_1} \begin{pmatrix} |a| \\ |b|e^{i\alpha} \\ |c|e^{i\alpha'} \end{pmatrix}, \quad (3)$$

where normalization condition is the following: $|a|^2 + |b|^2 + |c|^2 = 1$. Hence four independent parameters, for instance $|a|$, $|b|$, α , and α' define the quantum state. Therefore, having only two parameters in the Hamiltonian to change we cannot reach arbitrary quantum state starting from a given initial one, in other words, the evolution defined by the Hamiltonian (1)

cannot relate two arbitrary quantum states. In our restricted quantum brachistochrone problem we consider only the states which can be connected by the implemented pre-constrained evolution.

The evolution of the state vector can be realized as follows

$$|\psi(t)\rangle = e^{-iHt/\hbar}|\psi_i\rangle = e^{-i\omega(\mathbf{n}\cdot\mathbf{s})t}|\psi_i\rangle. \quad (4)$$

It is convenient to represent the unitary operator of evolution in the form

$$e^{-i\omega(\mathbf{n}\cdot\mathbf{s})t} = 1 - (\mathbf{n}\cdot\mathbf{s})^2 2 \sin^2 \frac{\omega t}{2} - i(\mathbf{n}\cdot\mathbf{s}) \sin \omega t. \quad (5)$$

In order to prove this let us note that $\mathbf{n}\cdot\mathbf{s}$ is the operator of the projection of spin-1 on the direction \mathbf{n} and it has three eigenvalue $-1, 0, 1$ with corresponding eigenvectors $|-1\rangle$, $|0\rangle$, $|1\rangle$ which can play the role of the basis vectors. An arbitrary vector of state can be written as linear combination of these vectors. It is enough to prove formula (5) only for basis vectors, which are eigenvectors of $\mathbf{n}\cdot\mathbf{s}$ with eigenvalues $-1, 0, 1$. It is easy to verify that for λ which takes only three values $-1, 0, 1$ we have

$$e^{\lambda x} = (1 - \lambda)(1 + \lambda) + \frac{1}{2}\lambda(\lambda + 1)e^x + \frac{1}{2}\lambda(\lambda - 1)e^{-x}. \quad (6)$$

Then using (6) for the unitary operator of evolution we just obtain (5).

Let us take the initial vector of state as the eigenvector of s_z with eigenvalue -1

$$|\psi_i\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad (7)$$

and reachable final state in form given by (3). Then using (4), and representation (5) for the operator of evolution, and matrix representation for spin in which s_z is diagonal, we finally find

$$|\psi(t)\rangle = \begin{pmatrix} -e^{-i2\phi} \sin^2 \theta \sin^2 \frac{\omega t}{2} \\ \sqrt{2}e^{-i\phi} \cos \theta \sin \theta \sin^2 \frac{\omega t}{2} - \frac{i}{\sqrt{2}}e^{-i\phi} \sin \theta \sin \omega t \\ 1 - (1 + \cos^2 \theta) \sin^2 \frac{\omega t}{2} + i \cos \theta \sin \omega t \end{pmatrix} \quad (8)$$

The first component gives the necessary condition that $|\psi(t)\rangle$ reaches the final state

$$\sin^2 \theta \sin^2 \frac{\omega t}{2} = |a|. \quad (9)$$

From (8) it follows that the second component depends on the first one. Substituting $\sin^2 \frac{\omega t}{2}$ from (9) into the second component of (8) we have

$$|b|^2 = 2|a|(1 - |a|). \quad (10)$$

Then the normalization condition yields the relation

$$|c|^2 = 1 - |a|^2 - |b|^2 = (1 - |a|)^2. \quad (11)$$

Thus we cannot reach arbitrary state, but only such ones which have components satisfying conditions (10) and (11). In addition note that the phases for the second and the third components are not independent but are related according to (8). If all necessary conditions are satisfied, then the time of evolution from the initial state to the allowed final one can be found from (9)

$$t_f = \frac{4}{\Delta\omega} \arcsin \left(\frac{\sqrt{|a|}}{\sin \theta} \right), \quad (12)$$

where $\sin \theta > \sqrt{|a|}$, $\hbar\Delta\omega = 2\hbar\omega$ is the distance between largest and smallest energy levels.

It is interesting to note that this expression is very similar to corresponding one for spin-1/2 (see, for instance, [6], Eq. (5)). The difference is that (12) contains $\sqrt{|a|}$ instead of $|a|$ and is two times larger (in [6] a is denoted as b and $\Delta\omega$ is denoted as ω).

We obtain the minimal time which just corresponds to the quantum brachistochrone for $\theta = \pi/2$, when magnetic field is perpendicular to z -axis

$$t_{\min} = \frac{4}{\Delta\omega} \arcsin \left(\sqrt{|a|} \right). \quad (13)$$

As an explicit example let us consider the case $|a| = 1$, then $|b| = |c| = 0$ and the final state

$$|\psi_f\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad (14)$$

is the eigenvector of s_z with the eigenvalue 1 and is orthogonal to the initial one (7). In this case we have the solution for the time only when $\theta = \pi/2$. Thus initial state (7) evolves to final one (14) only when magnetic field is perpendicular to z -axis. For time of evolution we have $t_f = t_{\min} = 2\pi/\Delta\omega$. This time is two times longer than the shortest possible

time obtained in [1, 2]. Note that the state vector describing the evolution in our case is not superposition of initial and final states. Therefore, it is not strange that the time of evolution is longer than minimal possible one.

Let us consider the next example with the initial state

$$|\psi_i\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \quad (15)$$

Now the evolution is given by the state vector

$$|\psi(t)\rangle = \begin{pmatrix} -\frac{1}{\sqrt{2}}e^{-i\phi} \left(2 \cos \theta \sin \theta \sin^2 \frac{\omega t}{2} + i \sin \theta \sin \omega t \right) \\ 1 - 2 \sin^2 \theta \sin^2 \frac{\omega t}{2} \\ \frac{1}{\sqrt{2}}e^{i\phi} \left(2 \cos \theta \sin \theta \sin^2 \frac{\omega t}{2} - i \sin \theta \sin \omega t \right) \end{pmatrix} = \begin{pmatrix} -z^* \\ 1 - 2\gamma \\ z \end{pmatrix}, \quad (16)$$

where

$$z = \frac{1}{\sqrt{2}}e^{i\phi} \left(2 \cos \theta \sin \theta \sin^2 \frac{\omega t}{2} - i \sin \theta \sin \omega t \right) = |z|e^{i(\phi-\alpha)}, \quad (17)$$

$$|z|^2 = 2\gamma(1-\gamma), \quad \gamma = \sin^2 \theta \sin^2 \frac{\omega t}{2}, \quad \tan \alpha = \frac{\cos(\omega t/2)}{\cos \theta \sin(\omega t/2)}.$$

Finally the evolution of the state vector can be represented in the form

$$|\psi(t)\rangle = (1-2\gamma) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \sqrt{2\gamma(1-\gamma)} \begin{pmatrix} e^{-i(\phi-\alpha)} \\ 0 \\ e^{i(\phi-\alpha)} \end{pmatrix}, \quad (18)$$

where γ and α are functions of time as given in (17). Let us consider the final state

$$|\psi_f\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \quad (19)$$

which is orthogonal to the initial one (15). In order to reach this state we put $\gamma(t_f) = 1/2$.

This condition gives us the time of evolution

$$t_f = \frac{4}{\Delta\omega} \arcsin \left(\frac{1}{\sqrt{2} \sin \theta} \right). \quad (20)$$

Then choosing additionally that $\phi = \alpha(t_f)$ we find that $|\psi(t_f)\rangle = |\psi_f\rangle$.

For $\theta = \pi/2$ we obtain the minimal time of evolution

$$t_{\min} = \frac{\pi}{\Delta\omega}. \quad (21)$$

It is interesting to note that this time is equal to the minimal possible time which can be obtained according to the statement of [1, 2] where *the state vector of evolution for minimal possible time belongs to the subspace spanned by the initial and final state vectors* or, in other words, the vector of evolution for minimal possible time is a superposition of initial and final states. In our example, as we see from (18), the state vector during evolution does not stay in the subspace spanned by the initial and final state vectors. Therefore, we can conclude that *in order to achieve the minimal possible time it is not necessary that during the evolution state vector lies all the time in the subspace spanned by the initial and final state vectors*. A pre-constrained family of Hamiltonians can yield the more complicated evolution, but still with the optimal time, what can have practical value for experiment. There is no contradiction with the results of [2, 3], the present result is in agreement with the fact that to stay on the subspace spanned on the initial and final states one should use the full freedom in the general family of Hamiltonians for the system under consideration, but what is important, it shows also that staying on the mentioned subspace, is not crucial for implementing the shortest time evolution.

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